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Fractional Exponential Function and Its Application

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Abstract: **In this paper, we obtain the limit representation of fractional exponential function. The application of fractional exponential function is illustrated. The main methods used in this paper are fractional L'Hospital's rule, Jumarie's modified Riemann-Liouville (R-L) fractional derivative, and a new multiplication of fractional analytic functions. In fact, our results are generalizations of the results of classical calculus.**

Keyword: **limit representation, fractional exponential function, fractional L'Hospital's rule, Jumarie's modified R-L fractional derivative, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is a natural generalization of calculus. Fractional calculus has long attracted the attention of scientists and engineers, and has been widely used in physics, control engineering, mechanics, dynamics, modeling, biology, economics, signal processing, electrical engineering and other fields [1-10].

However, the definition of fractional derivative is not unique, there are many useful definitions, including Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [11-15]. Jumarie revised the definition of R-L fractional derivative with a new formula, and we obtained that the modified fractional derivative of a constant function is zero. Therefore, by using this definition, it is easier to associate fractional calculus with classical calculus.

In this paper, we get the limit representation of fractional exponential function. The methods used in this paper are fractional L'Hospital's rule, Jumarie type of R-L fractional derivative, and a new multiplication of fractional analytic functions. On the other hand, an example is provided to illustrate the application of fractional exponential function. And our results are extensions of the results of traditional calculus.

II. PRELIMINARIES

First, the fractional calculus and some important properties used in this paper are introduced below.

Definition 2.1 ([16])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. The Jumarie's modified R-L α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt, \tag{1}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([17])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([18])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Definition 2.4 ([19])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{5}$$

Then

$$\begin{aligned} &f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} &f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{7}$$

Definition 2.5 ([20])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{8}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(g_\alpha(x^\alpha))^{\otimes k}, \tag{10}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}(f_\alpha(x^\alpha))^{\otimes k}. \tag{11}$$

Definition 2.6 ([20])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{12}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([20])**:** If $0 < \alpha \leq 1$, and x is any real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{13}$$

And $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2k}, \tag{14}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes(2k+1)}. \tag{15}$$

Definition 2.8: Let n be a positive integer, if $f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$. On the other hand, $\left(f_\alpha(x^\alpha)\right)^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the nth power of the fractional analytic function $f_\alpha(x^\alpha)$. And $\left(f_\alpha(x^\alpha)\right)^{\otimes -n} = \left(f_\alpha(x^\alpha)\right)^{\otimes -1} \otimes \cdots \otimes \left(f_\alpha(x^\alpha)\right)^{\otimes -1}$ is the nth power of $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$. In addition, we define $\left(f_\alpha(x^\alpha)\right)^{\otimes 0} = 1$.

Definition 2.9 ([21])**:** If $0 < \alpha \le 1$. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be α-fractional analytic functions. Then the α-fractional analytic function $f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)}$ is defined by

$$f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)} = E_\alpha\left(g_\alpha(x^\alpha) \otimes Ln_\alpha\left(f_\alpha(x^\alpha)\right)\right). \tag{16}$$

Theorem 2.10 (fractional L'Hospital's rule)**:** *If* $0 < \alpha \le 1$, c *is a real number, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$, $[g_\alpha(x^\alpha)]^{\otimes -1}$ *are* α-*fractional analytic functions at* c. *If* $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = 0$, *or* $\lim_{x\to c} f_\alpha(x^\alpha) = \pm\infty$, $\lim_{x\to c} g_\alpha(x^\alpha) = \pm\infty$, $\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c) \neq 0$, $\lim_{x\to c} f_\alpha(x^\alpha) \otimes [g_\alpha(x^\alpha)]^{\otimes -1}$ *and* $\lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}$ *exist. Then*

$$\lim_{x\to c} f_\alpha(x^\alpha) \otimes [g_\alpha(x^\alpha)]^{\otimes -1} = \lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}. \tag{17}$$

III. RESULT AND EXAMPLE

The following is the main result in this paper, we obtain the limit representation of fractional exponential function. **Theorem 3.1:** *Let* $0 < \alpha \le 1$, *and* n *is any positive integer, then*

$$E_\alpha(x^\alpha) = \lim_{n\to\infty} \left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes n}. \tag{18}$$

Proof Since

$$Ln_\alpha\left(\lim_{n\to\infty}\left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes n}\right)$$

$$= \lim_{n\to\infty} Ln_\alpha\left(\left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes n}\right)$$

$$= \lim_{n\to\infty} n \cdot Ln_\alpha\left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)$$

$$= \lim_{n\to\infty} \frac{Ln_\alpha\left(\left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -1} \otimes -\frac{1}{n^2} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -1} \otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha$$

$$= \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{19}$$

It follows that $\lim_{n\to\infty} \left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes n}$ is the inverse function of $Ln_\alpha(x^\alpha)$, that is,

$$\lim_{n\to\infty} \left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes n} = E_\alpha(x^\alpha).$$

Q.e.d.

Next , we give an example to illustrate the application of fractional exponential function.

Example 3.2: Let $0 < \alpha \leq 1$. Find the limit $\lim_{x \to 0^+} \left[\frac{1}{\Gamma(\alpha+1)} x^\alpha\right]^{\otimes(1-cos_\alpha(x^\alpha))}$.

Solution Since

$$\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha\right]^{\otimes(1-cos_\alpha(x^\alpha))} = E_\alpha\left(\left(1-cos_\alpha(x^\alpha)\right)\otimes Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)\right). \tag{20}$$

And

$$\lim_{x \to 0^+}\left(\left(1-cos_\alpha(x^\alpha)\right)\otimes Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)\right)$$

$$= \lim_{x \to 0^+}\left(\left[\left(1-cos_\alpha(x^\alpha)\right)^{\otimes -1}\right]^{\otimes -1} \otimes Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)\right)$$

$$= \lim_{x \to 0^+}\left(\left[-sin_\alpha(x^\alpha)\otimes\left(1-cos_\alpha(x^\alpha)\right)^{\otimes -2}\right]^{\otimes -1} \otimes \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -1}\right) \quad \text{(by fractional L'Hospital's rule)}$$

$$= \lim_{x \to 0^+}\left(\left(1-cos_\alpha(x^\alpha)\right)^{\otimes 2}\otimes\left(-\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes sin_\alpha(x^\alpha)\right)^{\otimes -1}\right)$$

$$= \lim_{x \to 0^+}\left(2\cdot sin_\alpha(x^\alpha)\otimes\left(1-cos_\alpha(x^\alpha)\right)\otimes\left[-sin_\alpha(x^\alpha)-\frac{1}{\Gamma(\alpha+1)} x^\alpha\otimes cos_\alpha(x^\alpha)\right]^{\otimes -1}\right)$$

(again by fractional L'Hospital's rule)

$$= \lim_{x \to 0^+}\left(\left[2\cdot sin_\alpha(x^\alpha)-sin_\alpha(2x^\alpha)\right]\otimes\left[-sin_\alpha(x^\alpha)-\frac{1}{\Gamma(\alpha+1)} x^\alpha\otimes cos_\alpha(x^\alpha)\right]^{\otimes -1}\right)$$

$$= \lim_{x \to 0^+}\left(\left[2\cdot cos_\alpha(x^\alpha)-2\cdot cos_\alpha(2x^\alpha)\right]\otimes\left[-2\cdot cos_\alpha(x^\alpha)+\frac{1}{\Gamma(\alpha+1)} x^\alpha\otimes sin_\alpha(x^\alpha)\right]^{\otimes -1}\right)$$

(again by fractional L'Hospital's rule)

$$= 0.$$

It follows that

$$\lim_{x \to 0^+}\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha\right]^{\otimes(1-cos_\alpha(x^\alpha))}$$

$$= \lim_{x \to 0^+} E_\alpha\left(\left(1-cos_\alpha(x^\alpha)\right)\otimes Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)\right)$$

$$= E_\alpha\left(\lim_{x \to 0^+}\left(\left(1-cos_\alpha(x^\alpha)\right)\otimes Ln_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)\right)\right)$$

$$= E_\alpha(0)$$

$$= 1. \tag{21}$$

IV. CONCLUSION

In this paper, the limit representation of fractional exponential function is obtained. In addition, we provide an example to illustrate the application of fractional exponential function. Fractional L'Hospital's rule and a new multiplication of fractional analytic functions play important roles in this paper. In fact, the results obtained in this paper are the extension of the results of traditional calculus. In the future, we will continue to use fractional exponential function to study the problems in fractional differential equations and engineering mathematics.

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